Hindawi
Advances in Civil Engineering
Volume 2019, Article ID 1560171, 12 pages
https://doi.org/10.1155/2019/1560171

Research Article

Deformation and Mechanical Behaviors of SCSF and CCCF Rectangular Thin Plates Loaded by Hydrostatic Pressure

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Received 28 December 2018; Accepted 6 February 2019; Published 7 March 2019

Academic Editor: Harry Far

Elastic rectangular thin plate problems are very important both in theoretical research and engineering applications. Based on this, the flexural deformation functions of the rectangular thin plates with two opposite edges simply supported, one edge clamped and one edge free (SCSF) and three edges clamped and one edge free (CCCF), loaded by hydrostatic pressure are determined by single trigonometric series. And the flexural deformation functions are solved via the principle of minimum potential energy. Next, the internal force and stress functions of rectangular thin plates with two boundary conditions are obtained based on the small deflection bending theory of thin plates. The dimensionless deflection, dimensionless internal force, and dimensionless stress functions of the rectangular thin plates are established as well. The analytic solution in this paper is validated by the finite element method. Finally, the influence of aspect ratio λ and Poisson's ratio μ on the deformation and mechanical behaviors of the rectangle thin plates is analyzed in this paper. This research can provide references for the plane water gate problem in seaports and channels.

1. Introduction

Bending of rectangular thin plates has been heavily researched and gained great achievements. Numerical and analytical methods are two research methods that are often used for the analysis of thin plate problems. It is well known that many effective numerical methods have been developed in recent years. Representative methods include the finite element method (FEM) [1, 2], the finite difference method [3–5], the finite strip method [6], the meshless method [7], the spline element method [8], etc. These numerical methods normally meet the engineering requirements with acceptable errors and are greatly applied in practice. Meanwhile, analytical solutions are regarded as the benchmarks for verification of various numerical methods and have been investigated by many researchers. The problem of a rectangular thin plate is first given by Dixon [9]. The solution of double trigonometric series of a rectangular thin plate with four edges simply supported (SSSS) under arbitrary loading has been proposed by Navier (Navier's solution). The solution of single trigonometric series of a rectangular thin plate with two opposite edges simply supported and other two opposite edges free (SFSF) under transverse loading has been given by Levy (Levy's solution). Different methods are used to analyze the rectangular thin plate problem under different boundary conditions and loading, such as the Fourier series method, the Rayleigh–Ritz method, the superposition method, the semi-inverse method, the symplectic geometry method, the integral-transform method, etc. The bending problem of a square plate with two adjacent edges clamped and the others either simply supported or free (CCSS or CCFF) under uniform loading has been investigated by Huang and Conway [10]. Many exact solutions for the bending problem of elastic rectangular thin plates have been obtained by Timoshenko and Woinwsky-Krieger [11]. The bending problem of a rectangular thin plate with two opposite edges simply supported has been analyzed by Hutchinson [12]. The buckling problem of clamped rectangular plates with different aspect ratios has been solved by El-Bayoumy using extended Kantorovich method [13]. The problem of an isotropic rectangular thin plate with four edges clamped has been given by Imrak and Gerdemeli [14]. The accurate solution for a rectangular thin plate with two adjacent edges clamped and the others free (CCFF) is proposed by Chang [15], and the solution is yielded by superposing six known solutions. The symplectic geometry method is employed by Lim et al. [16] to investigate the bending problem of a rectangular thin plate with two opposite edges simply supported and the others free (SFSF). Moreover, the symplectic geometry method is also employed by Zhong and Li [17] and Liu and Li [18] to solve the deflection function and bending moment of a rectangular thin plate with four edges clamped (CCCC) under arbitrary loading. The analytic bending solutions of free rectangular thin plates resting on elastic foundations are obtained by Li et al. [19] via a new accurate symplectic superposition method. Moreover, Li et al. [20] extend the approach to the free vibration problems of the same plates and obtain the analytic solutions which cannot be obtained by the conventional symplectic approach. Besides, the analytic bending solutions of rectangular thin plates with a corner point-supported, its adjacent corner free, and their opposite edge clamped or simply supported are obtained by Li et al. [21] via the superposition method in the symplectic space. The method of symplectic geometry is more reasonable than traditional semi-inverse solution. Khan et al. [22] employed the variation method to obtain a higher approximate solution for a rectangular thin plate with four edges simply supported (SSSS) under uniform loading. Based on this, the bending problem of rectangular thin plates has also been investigated by some other researchers under different boundary conditions and loading [23–26]. However, it is still difficult to obtain the exact solution through solving the differential equation for the bending problem of the rectangular thin plate with certain boundary conditions. Thus, many exact solutions for the bending of thin plates are obtained with simple boundary conditions and transverse loading, such as the bending of thin plates with four edges clamped or simply supported, two opposite edges clamped or simply supported, three edges clamped or simply supported under uniform loading, and transverse loading. Most approximate solutions have been obtained for the bending problem of rectangular thin plates with relatively complicated boundary conditions and transverse loading.

In this paper, the flexural deformation functions of two types of rectangular thin plates (two opposite edges simply supported, one edge clamped and one edge free (SCSF) and three edges clamped and one edge free (CCCF)) loaded by hydrostatic pressure are established with single trigonometric series. The flexural deformation functions are solved using the principle of minimum potential energy. Internal force and stress functions of the rectangular thin plates under the two boundary conditions are obtained using the small deflection bending theory of thin plates. The dimensionless deflection, dimensionless internal force, and dimensionless stress functions of rectangular thin plates under the two boundary conditions are established in this paper. Moreover, the influence of aspect ratio and Poisson's ratio on the deformation and mechanical characteristics of rectangular thin plates under the two boundary conditions is analyzed in this paper. This research can provide references for the plane water gate problem of seaports and channels.

2. Deflection and Internal Force Function of the SCSF Rectangular Thin Plate

2.1. Bending Equation and Boundary Condition of the SCSF Rectangular Thin Plate. The hydrostatic pressure $q_w = q_0(1 - y/b)$ is loaded on the surface of the rectangular thin plate. The width is a along the x axis. The height is b along the y axis. The thickness is δ along the z axis. The dimensions and load condition of the rectangular thin plate are shown in Figure 1.

The governing differential equation for the bending problem of the rectangular thin plate is as follows:

$$D\nabla^4 w(x, y) = q(x, y), \tag{1}$$

where $D = E\delta^3/12(1 - v^2)$ is the flexural rigidity. E, δ, v are the elastic modulus, plate thickness, and Poisson's ratio, respectively. $w(x, y)$ is the transverse deflection. $q(x, y)$ is the distributed transverse load acting on the surface of the plate. $(0 \le x \le a,\ 0 \le y \le b,\ -\delta/2 \le z \le \delta/2)$.

The edges of $x = 0$ and $x = a$ are simply supported, $y = 0$ is clamped, and $y = b$ is free. The boundary condition of the SCSF rectangular thin plate can be expressed as follows:

$$\begin{aligned} w|_{x=0} &= 0, \\ w|_{x=a} &= 0, \\ w|_{y=0} &= 0, \\ \left.\frac{\partial w}{\partial y}\right|_{y=0} &= 0. \end{aligned} \tag{2}$$

It is difficult to obtain the deflection function if we solve the differential equation for the bending problem of the rectangular thin plate directly with the boundary conditions. Thus, the deflection of thin plates is solved via Rayleigh–Ritz method.

2.2. Flexural Function. Based on the small deflection assumption for the thin plate-bending problems, the deflection w is the only unknown function, and other components can be expressed in terms of w. The expression of deflection w can be expressed as $w = \sum_{m=1,3,5,\ldots}^{\infty} C_m w_m$, where C_m is the independent and undetermined coefficient and w_m is the deflection function. The deflection function of the SCSF rectangular thin plate loaded by hydrostatic pressure is as follows:

$$w(x, y) = \sum_{m=1,3,5,\ldots}^{\infty} C_m w_m = \sum_{m=1,3,5,\ldots}^{\infty} C_m \sin\left(\frac{m\pi x}{a}\right)\left(\frac{y}{b}\right)^2. \tag{3}$$

The deflection function satisfies the boundary conditions of equation (2), where C_m is the undetermined constant. The expression for strain energy of the thin plate is as follows:

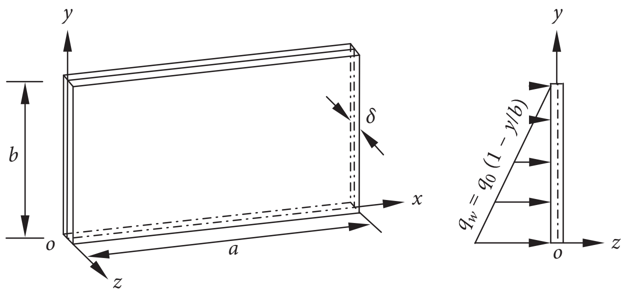

Figure 1: Rectangular thin plate under hydrostatic pressure.

$$V_\varepsilon = \frac{D}{2}\iint_A \left\{ \left(\nabla^2 w\right)^2 - 2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right\} dx\,dy, \tag{4}$$

where A is the area of the thin plate, $\nabla^2 w = (\partial^2 w/\partial x^2) + (\partial^2 w/\partial y^2)$.

Solving the second derivative of the deflection function w versus x and y, respectively, and substituting them into equation (4), the expression for strain energy of thin plate can be written as follows:

$$V_\varepsilon = \sum_{m=1,3,5,\ldots}^{\infty} \frac{DC_m^2}{2}\left[2 + \left(\frac{4}{3} - 2\nu\right)\left(\frac{m\pi b}{a}\right)^2 + \frac{1}{10}\left(\frac{m\pi b}{a}\right)^4\right]\frac{a}{b^3}. \tag{5}$$

The first derivative of the strain energy V_ε versus the coefficient C_m is as follows:

$$\frac{\partial V_\varepsilon}{\partial C_m} = DC_m\left[2 + \left(\frac{4}{3} - 2\nu\right)\left(\frac{m\pi b}{a}\right)^2 + \frac{1}{10}\left(\frac{m\pi b}{a}\right)^4\right]\frac{a}{b^3}. \tag{6}$$

From equation (3), we get

$$\begin{aligned}\iint_A q w_m\,dx\,dy &= \iint_A q_0\left(1 - \frac{y}{b}\right) w_m\,dx\,dy \\ &= \int_0^a\int_0^b q_0\left(1 - \frac{y}{b}\right)\left(\frac{y}{b}\right)^2 \sin\frac{m\pi x}{a}\,dx\,dy \\ &= \frac{q_0 ab}{6m\pi}.\end{aligned} \tag{7}$$

Based on the principle of minimum potential energy, the first derivative of the strain energy V_ε versus the coefficient C_m can be expressed as follows:

$$\frac{\partial V_\varepsilon}{\partial C_m} = \int_0^a\int_0^b q w_m\,dx\,dy. \tag{8}$$

Substituting equations (6) and (7) into equation (8), the coefficient equation can be written as

$$C_m = \frac{q_0 b^4}{6m\pi D\left[2 + ((4/3) - 2\nu)(m\pi b/a)^2 + (1/10)(m\pi b/a)^4\right]}. \tag{9}$$

Substituting equation (9) into equation (3) allows the deflection function w to be written as

$$w = \sum_{m=1,3,5,\ldots}^{\infty}\left\{\frac{q_0 b^2 \sin(m\pi x/a) y^2}{6m\pi D\left[2 + ((4/3) - 2\nu)(m\pi b/a)^2 + (1/10)(m\pi b/a)^4\right]}\right\}. \tag{10}$$

The dimensionless deflection $w' = Dw/q_0 b^4$ can be formulated as

$$w' = \sum_{m=1,3,5,\ldots}^{\infty}\left\{\frac{\sin m\pi x' \cdot y'^2}{6m\pi\left[2 + ((4/3) - 2\nu)\left(m^2\pi^2/\lambda^2\right) + \left(m^4\pi^4/10\lambda^4\right)\right]}\right\}, \tag{11}$$

where $x' = x/a$, $y' = y/b$, and $\lambda = a/b$.

2.3. Dimensionless Internal Force Function and Stress Function. Substituting the deflection function w of the SCSF rectangular thin plate loaded by the hydrostatic pressure into the internal force equations and stress equations of classical elastic thin plate, the internal force equation and stress equation can be rewritten as

$$\left.\begin{aligned} M_x &= \sum_{m=1,3,5,\ldots}^{\infty} -C_m D\frac{2\nu a^2 - m^2\pi^2 y^2}{a^2 b^2}\sin\frac{m\pi x}{a} \\ M_y &= \sum_{m=1,3,5,\ldots}^{\infty} -C_m D\frac{2a^2 - \nu m^2\pi^2 y^2}{a^2 b^2}\sin\frac{m\pi x}{a} \\ M_{xy} &= \sum_{m=1,3,5,\ldots}^{\infty} -C_m D\frac{2(1-\nu)m\pi}{ab^2}\cos\frac{m\pi x}{a}y \end{aligned}\right\},$$

$$\left.\begin{aligned} F_{Sx} &= \sum_{m=1,3,5,\ldots}^{\infty} -C_m D\frac{m\pi\left(2a^2 - m^2\pi^2 y^2\right)}{a^3 b^2}\cos\frac{m\pi x}{a} \\ F_{Sy} &= \sum_{m=1,3,5,\ldots}^{\infty} 2C_m D\left(\frac{m\pi}{ab}\right)^2\sin\frac{m\pi x}{a}y \end{aligned}\right\},$$

$$\left.\begin{aligned} \sigma_x &= \sum_{m=1,3,5,\ldots}^{\infty} -\frac{12C_m D\left(2\nu a^2 - m^2\pi^2 y^2\right)}{a^2 b^2\delta^3}\sin\frac{m\pi x}{a}z \\ \sigma_y &= \sum_{m=1,3,5,\ldots}^{\infty} -\frac{12C_m D\left(2a^2 - \nu m^2\pi^2 y^2\right)}{a^2 b^2\delta^3}\sin\frac{m\pi x}{a}z \\ \tau_{xy} &= \sum_{m=1,3,5,\ldots}^{\infty} -\frac{24C_m D(1-\nu)m\pi}{ab^2\delta^3}\cos\frac{m\pi x}{a}yz \end{aligned}\right\}. \tag{12}$$

Moreover, the dimensionless bending moments $M'_{ij} = a^2 M_{ij}/q_0 b^4$, dimensionless shear forces $F'_{Si} = a^3 F_{Si}/q_0 b^4$, and dimensionless stresses $\sigma'_{ij} = a^2\delta^2\sigma_{ij}/q_0 b^4$ can be established as follows:

$$\left.\begin{aligned} M'_x &= \sum_{m=1,3,5,\ldots}^{\infty} -C'_m D\left(2\nu\lambda^2 - m^2\pi^2 y'^2\right)\sin m\pi x' \\ M'_y &= \sum_{m=1,3,5,\ldots}^{\infty} -C'_m D\left(2\lambda^2 - \nu m^2\pi^2 y'^2\right)\sin m\pi x' \\ M'_{xy} &= \sum_{m=1,3,5,\ldots}^{\infty} -2C'_m D(1-\nu)m\pi\lambda\cos m\pi x'\cdot y' \end{aligned}\right\},$$
$$\left.\begin{aligned} F'_{Sx} &= \sum_{m=1,3,5,\ldots}^{\infty} -C'_m Dm\pi\left(2\lambda^2 - m^2\pi^2 y'^2\right)\cos m\pi x' \\ F'_{Sy} &= \sum_{m=1,3,5,\ldots}^{\infty} 2C'_m Dm^2\pi^2\lambda\sin m\pi x'\cdot y' \end{aligned}\right\},$$
$$\left.\begin{aligned} \sigma'_x &= \sum_{m=1,3,5,\ldots}^{\infty} -12C'_m D\left(2\nu\lambda^2 - m^2\pi^2 y'^2\right)\sin m\pi x'\cdot z' \\ \sigma'_y &= \sum_{m=1,3,5,\ldots}^{\infty} -12C'_m D\left(2\lambda^2 - \nu m^2\pi^2 y'^2\right)\sin m\pi x'\cdot z' \\ \tau'_{xy} &= \sum_{m=1,3,5,\ldots}^{\infty} -24C'_m D(1-\nu)m\pi\lambda\cos m\pi x'\cdot y'z' \end{aligned}\right\}, \tag{13}$$

where M_{ij} is the bending moment, F_{Si} is the shear force, σ_{ij} is the stress tensor, $z' = z/\delta$, $C'_m = C_m/q_0 b^4$, and $i, j = x, y$.

3. Deflection and Internal Force Function of the CCCF Rectangular Thin Plate

Similarly, the flexural deformation function of the CCCF rectangular thin plate loaded by hydrostatic pressure can also be established by single trigonometric series. The flexural deformation function coefficient is solved using the Rayleigh–Ritz method and the principle of minimum potential energy. The internal force and stress functions are obtained via the small deflection bending theory of thin plate.

3.1. Bending Equation and Boundary Condition of the CCCF Rectangular Thin Plate. The rectangular thin plate surface is loaded by the hydrostatic pressure $q_w = q_0(1 - y/b)$ (along the direction of y). The width, height, and thickness are shown in Figure 1. The governing differential equation for the bending problem of rectangular thin plate is given in equation (1). The edges of $x = 0$, $x = a$, and $y = 0$ are clamped and $y = b$ is free. The boundary condition of the CCCF rectangular thin plate can be expressed as follows:

$$\begin{aligned} w|_{x=0,a} &= 0, \\ \left.\frac{\partial w}{\partial x}\right|_{x=0,a} &= 0, \\ w|_{y=0} &= 0, \\ \left.\frac{\partial w}{\partial y}\right|_{y=0} &= 0. \end{aligned} \tag{14}$$

3.2. Flexural Function. Based on the small deflection assumption for thin plate-bending problems, the deflection w is the only unknown function, and other components can be expressed in terms of w. The expression of deflection w is $w = \sum_{m=1}^{\infty} C_m w_m$, where C_m is the undetermined coefficients and w_m is the deflection function. The deflection function of the CCCF rectangular thin plate loaded by the hydrostatic pressure is as follows:

$$w(x,y) = \sum_{m=1}^{\infty} C_m w_m = \sum_{m=1}^{\infty} C_m \sin^2\left(\frac{m\pi x}{a}\right)\left(\frac{y}{b}\right)^2. \tag{15}$$

Thus, the deflection function satisfies the boundary condition of equation (14). Solving the second derivative of the deflection function w versus x and y and substituting them into equation (4), the expression for strain energy of the thin plate can be rewritten as follows:

$$V_\varepsilon = \sum_{m=1}^{\infty} \frac{DC_m^2}{2}\left[\frac{3}{2} + \left(\frac{4}{3} - 2\nu\right)\left(\frac{m\pi b}{a}\right)^2 + \frac{2}{5}\left(\frac{m\pi b}{a}\right)^4\right]\frac{a}{b^3}. \tag{16}$$

The first derivative of the strain energy V_ε versus C_m can be deduced as follows:

$$\frac{\partial V_\varepsilon}{\partial C_m} = DC_m\left[\frac{3}{2} + \left(\frac{4}{3} - 2\nu\right)\left(\frac{m\pi b}{a}\right)^2 + \frac{2}{5}\left(\frac{m\pi b}{a}\right)^4\right]\frac{a}{b^3}. \tag{17}$$

From equation (15), we get

$$\begin{aligned} \int_0^a\int_0^b q w_m\,dx\,dy &= \iint_A q_0\left(1 - \frac{y}{b}\right) w_m\,dx\,dy \\ &= \int_0^a\int_0^b q_0\left(1 - \frac{y}{b}\right)\left(\frac{y}{b}\right)^2 \sin^2\frac{m\pi x}{a}\,dx\,dy \\ &= \frac{q_0 ab}{24}. \end{aligned} \tag{18}$$

Based on the principle of minimum potential energy equation (8), substituting equations (17) and (18) into equation (8) gives the expression of the coefficient C_m as follows:

$$C_m = \frac{q_0 b^4}{24D\left[(3/2) + ((4/3) - 2\nu)(m\pi b/a)^2 + (2/5)(m\pi b/a)^4\right]}. \tag{19}$$

Substituting equation (19) into equation (15), the deflection function w can be written as follows:

$$w = \sum_{m=1}^{\infty} \frac{q_0 b^2 \sin^2(m\pi x/a) y^2}{24\, D\left[(3/2) + ((4/3) - 2\nu)(m\pi b/a)^2 + (2/5)(m\pi b/a)^4\right]}. \tag{20}$$

The dimensionless deflection $w' = Dw/q_0 b^4$ is as follows:

$$w' = \sum_{m=1}^{\infty} \frac{\sin^2 m\pi x' \cdot y'^2}{24\left[(3/2) + ((4/3) - 2\nu)\left(m^2\pi^2/\lambda^2\right) + \left(2m^4\pi^4/5\lambda^4\right)\right]}. \tag{21}$$

3.3. Dimensionless Internal Force Function and Stress Function. Substituting the deflection function w of the CCCF rectangular thin plate loaded by hydrostatic pressure into the internal force equation and stress equation of elastic thin plate, the internal force equation and stress equation can be rewritten as follows:

$$\left.\begin{aligned}
M_x &= \sum_{m=1}^{\infty} -2C_m D\left(\frac{\nu}{b^2}\sin^2\frac{m\pi x}{a} + \frac{m^2\pi^2}{a^2 b^2}\cos\frac{2m\pi x}{a}y^2\right)\\
M_y &= \sum_{m=1}^{\infty} -2C_m D\left(\frac{1}{b^2}\sin^2\frac{m\pi x}{a} + \frac{\nu m^2\pi^2}{a^2 b^2}\cos\frac{2m\pi x}{a}y^2\right)\\
M_{xy} = M_{yx} &= \sum_{m=1}^{\infty} -2C_m D(1-\nu)\frac{m\pi}{ab^2}\sin\frac{2m\pi x}{a}y
\end{aligned}\right\},$$

$$\left.\begin{aligned}
F_{Sx} &= \sum_{m=1}^{\infty} -2C_m D\frac{m\pi\left(a^2 - 2m^2\pi^2 y^2\right)}{a^3 b^2}\sin\frac{2m\pi x}{a}\\
F_{Sy} &= \sum_{m=1}^{\infty} -4C_m D\frac{m^2\pi^2}{a^2 b^2}\cos\frac{2m\pi x}{a}y
\end{aligned}\right\},$$

$$\left.\begin{aligned}
\sigma_x &= \sum_{m=1}^{\infty} -\frac{24C_m D}{\delta^3}\left(\frac{\nu}{b^2}\sin^2\frac{m\pi x}{a} + \frac{m^2\pi^2}{a^2 b^2}\cos\frac{2m\pi x}{a}y^2\right)z\\
\sigma_y &= \sum_{m=1}^{\infty} -\frac{24C_m D}{\delta^3}\left(\frac{1}{b^2}\sin^2\frac{m\pi x}{a} + \frac{\nu m^2\pi^2}{a^2 b^2}\cos\frac{2m\pi x}{a}y^2\right)z\\
\tau_{xy} = \tau_{yx} &= \sum_{m=1}^{\infty} -\frac{24(1-\nu)m\pi C_m D}{ab^2\delta^3}\sin\frac{2m\pi x}{a}yz
\end{aligned}\right\}. \tag{22}$$

Then, the dimensionless bending moments $M'_{ij} = a^2 M_{ij}/q_0 b^4$, dimensionless shear forces $F'_{Si} = a^3 F_{Si}/q_0 b^4$, and dimensionless stresses $\sigma'_{ij} = a^2\delta^2\sigma_{ij}/q_0 b^4$ can be established as follows:

$$\left.\begin{aligned}
M'_x &= \sum_{m=1}^{\infty} -2C'_m D\left(\nu\lambda^2\sin^2 m\pi x' + m^2\pi^2 y'^2\cos 2m\pi x'\right)\\
M'_y &= \sum_{m=1}^{\infty} -2C'_m D\left(\lambda^2\sin^2 m\pi x' + \nu m^2\pi^2 y'^2\cos 2m\pi x'\right)\\
M'_{xy} = M'_{yx} &= \sum_{m=1}^{\infty} -2C'_m D(1-\nu)m\pi\lambda\sin 2m\pi x'\cdot y'
\end{aligned}\right\},$$

$$\left.\begin{aligned}
F'_{Sx} &= \sum_{m=1}^{\infty} -2C'_m Dm\pi\left(\lambda^2 - 2m^2\pi^2 y'^2\right)\sin 2m\pi x'\\
F'_{Sy} &= \sum_{m=1}^{\infty} -4C'_m D\lambda m^2\pi^2\cos 2m\pi x'\cdot y'
\end{aligned}\right\},$$

$$\left.\begin{aligned}
\sigma'_x &= \sum_{m=1}^{\infty} -24C'_m D\left(\nu\lambda^2\sin^2 m\pi x' + m^2\pi^2 y'^2\cos 2m\pi x'\right)z'\\
\sigma'_y &= \sum_{m=1}^{\infty} -24C'_m D\left(\lambda^2\sin^2 m\pi x' + \nu m^2\pi^2 y'^2\cos 2m\pi x'\right)z'\\
\tau'_{xy} = \tau'_{yx} &= \sum_{m=1}^{\infty} -24C'_m D(1-\nu)m\pi\lambda\sin 2m\pi x'\cdot y'z'
\end{aligned}\right\}. \tag{23}$$

4. Results and Discussion

The influence of aspect ratio λ (0.5, 1.0, 1.5, and 2.0) and Poisson's ratio μ (0.25, 0.30, and 0.35) on the deformation and mechanical properties of the rectangle thin plates with two boundary conditions is analyzed in this paper. The physical and mechanical parameters of the two types of rectangular thin plates are shown in Table 1. It is known that larger values of m give calculation accuracy. The distribution regularities of dimensionless deflection, dimensionless internal force, and dimensionless stress of the two rectangular thin plates loaded by the hydrostatic pressure are shown in Figures 2–7.

4.1. Influence of Aspect Ratio λ on the Deformation and Mechanical Properties of the Rectangular Thin Plates. The comparison between the analytic solution presented in this paper and the finite element method (FEM) via the software package ABAQUS for the SCSF rectangular thin plate with aspect ratio $\lambda = 2.0$ is shown in Figures 2 and 3. The finite element types used in ABAQUS are C3D8R and 22400 uniform elements. Figures 2 and 3 show that there are some errors between the analytical solutions presented in this paper and the numerical solutions obtained by the FEM. The errors are mainly caused by the values of m and the selection of the solution method. The larger the values of m, the closer the analytical solutions to the exact solutions and the smaller the error. Moreover, the flexural deformation function of the SCSF rectangular thin plate loaded by the hydrostatic pressure is obtained via the Rayleigh–Ritz method in this paper. It is known that the thin plate-bending problems are solved by the Rayleigh–Ritz method; only the flexural deformation function is required to meet the displacement boundary conditions, but

Table 1: Physical and mechanical parameters of the SCSF and CCCF rectangular thin plates.

Height (m)	Thickness (m)	Elastic modulus E (GPa)	Loading q_0 (kN)	Series item m
2.0	0.1	25.0	20.0	1.0

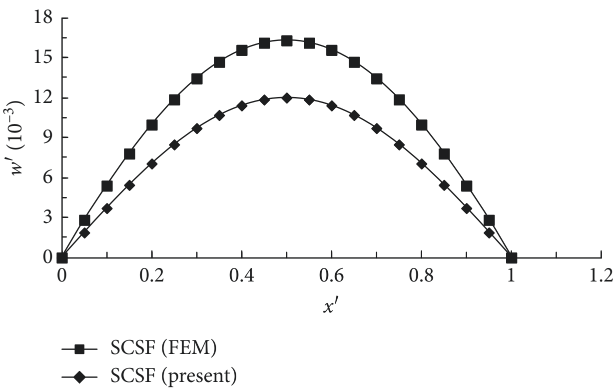

Figure 2: Relationship between dimensionless deflection and x' at y' = 1.0 (SCSF).

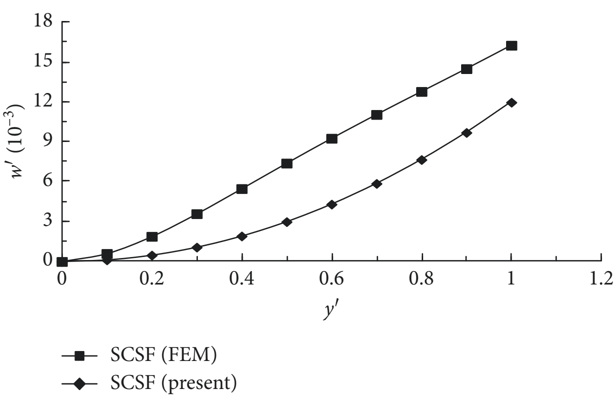

Figure 3: Relationship between dimensionless deflection and y' at x' = 0.5 (SCSF).

not the internal force boundary conditions (if it can partially or completely meet the internal force boundary conditions, the accuracy of solutions can be improved). Equation (2) is the displacement boundary conditions. Thus, the accuracy of the solutions can be improved via the larger values of m and the selection of the trial functions that meet the displacement boundary conditions and internal force boundary conditions.

As shown in Figures 2 and 3, it is obvious that the results of FEM computation well agree with that of the analytical computation, which demonstrates the correctness of the present method for the SCSF rectangular thin plate. The distribution regularities of the dimensionless deflection, dimensionless internal force, and dimensionless stress of the SCSF rectangular thin plate with different values of aspect ratio are given in Figure 4.

As shown in Figure 4, w', M_x', M_y', F_{Sy}', σ_x', and σ_y' of the SCSF rectangular thin plate are symmetrically distributed, which is divided by x' = 0.5, and M_{xy}', F_{Sx}', and τ_{xy}' are antisymmetrically distributed. The maximum of w', M_x', F_{Sy}', and σ_x' appears at the point of (0.5,1). F_{Sx}' is shown in four-quadrant chart, and the extremum of F_{Sx}' is appeared at the four corners of the SCSF rectangular thin plate. Besides, the horizontal axis moves upward with the increase of aspect ratio λ. The extremum of M_y', σ_y' appears at the points of (0.5, 0) and (0.5, 1), and the positive M_y', σ_y' gradually disappear with the increase of aspect ratio λ. The extremum of M_{xy}', τ_{xy}' appears at the points of (0, 1) and (1, 1). The maximums of dimensionless deflection, dimensionless internal force, and dimensionless stress of the SCSF rectangular thin plate loaded by the hydrostatic pressure with different values of aspect ratio are shown in Table 2.

The comparison between the analytic solution presented in this paper and the well-accepted finite element method (FEM) via software package ABAQUS for the CCCF rectangular thin plate with aspect ratio $\lambda = 2.0$ is shown in Figures 5 and 6. The finite element types used in ABAQUS are C3D8R and 22400 uniform elements. Similar to Figures 2 and 3, Figures 5 and 6 show that there are some errors between the analytical solutions and the numerical solutions. The errors are mainly caused by the values of m and the selection of the solution method. The larger the values of m, the closer the analytical solutions to the exact solutions and the smaller the error. Moreover, the flexural deformation function of the CCCF rectangular thin plate loaded by the hydrostatic pressure is obtained via the Rayleigh–Ritz method. It is known that the thin plate-bending problems are solved by the Rayleigh–Ritz method, only the flexural deformation function is required to meet the displacement boundary conditions, but not the internal force boundary conditions (if it can partially or completely meet the internal force boundary conditions, the accuracy of solutions can be improved). Equation (14) is the displacement boundary conditions. Thus, the accuracy of the solutions can be improved via the larger values of m and the selection of the trial functions that meet the displacement boundary conditions and internal force boundary conditions.

As shown in Figures 5 and 6, it is obvious that the FEM computation is almost equivalent to the analytical computation, which the correctness of the present method for the CCCF rectangular thin plate is verified. Thus, the present method of this research is viable. The distribution regularities of the dimensionless deflection, dimensionless internal force and dimensionless stress of the CCCF rectangular thin plate with different values of aspect ratio are given in Figure 7.

Similarly, as shown in Figure 7, w', M_x', M_y', F_{Sy}', σ_x', and σ_y' of the CCCF rectangular thin plate are symmetrically distributed (divided by $x' = 0.5$), and M_{xy}', F_{Sx}', and τ_{xy}' are also antisymmetrically distributed (divided by $x' = 0.5$). The maximum of w' is appeared at the points of (0.5 1). The extremum of M_x', M_y', σ_x' and σ_y' are appeared at the four corners of the SCSF rectangular thin plate, which is the points of (0.5, 0) and (0.5, 1). The extremum of F_{Sx}' is appeared at the points of (0.25, 0), (0.75, 0), (0.25, 1), and (0.75, 1). The extremum of F_{Sy}' is appeared at the points of (0,

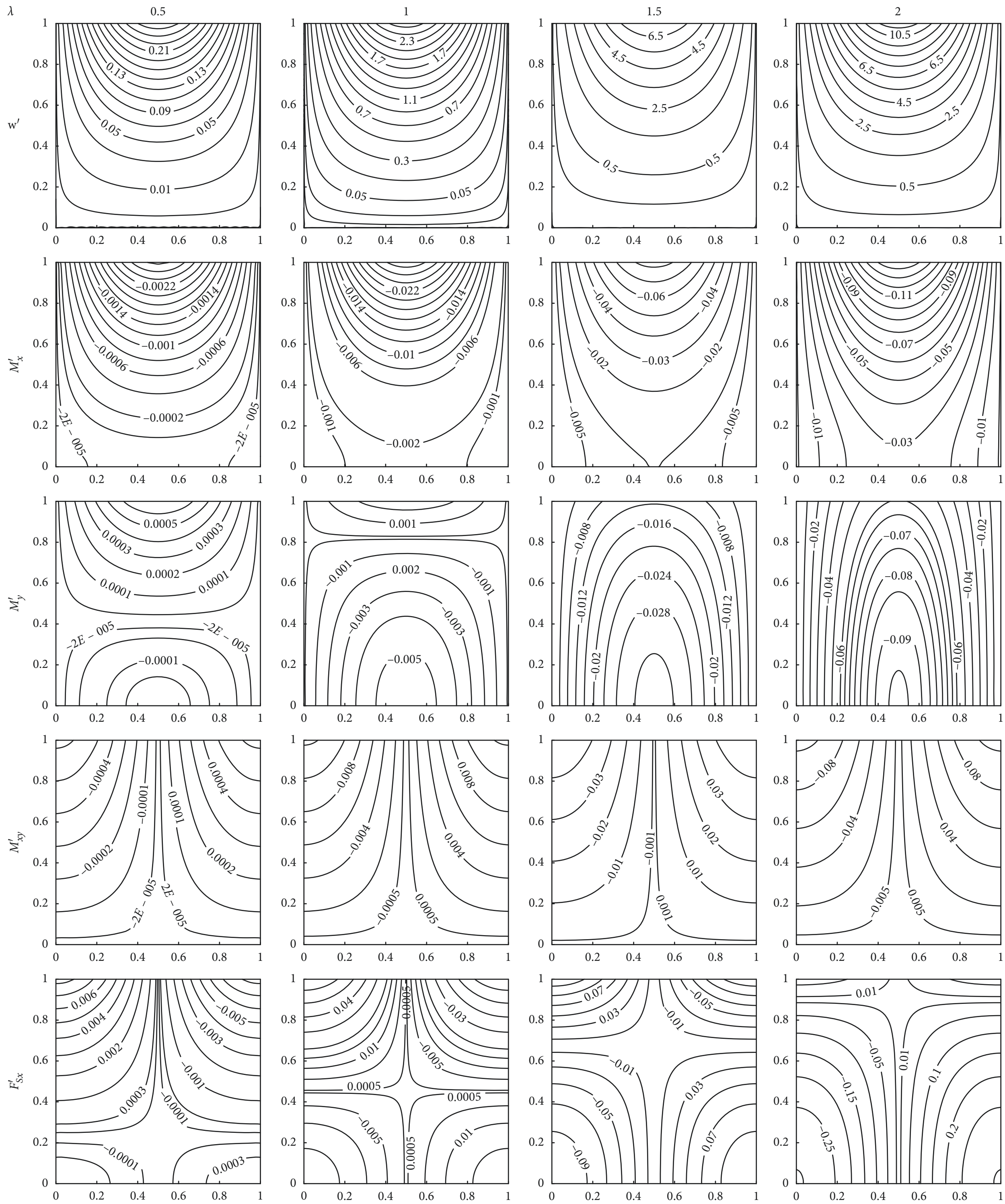

Figure 4: Continued.

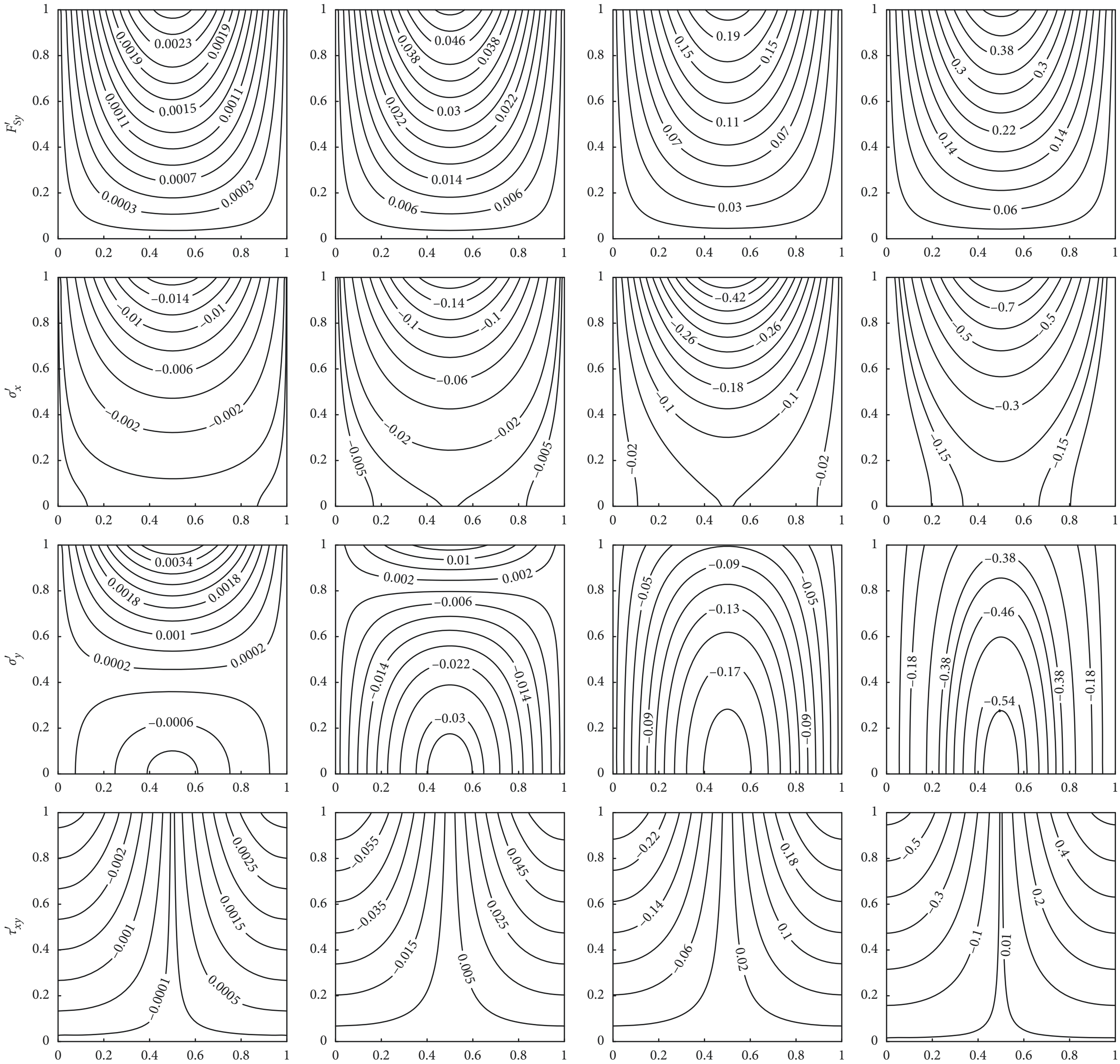

Figure 4: Dimensionless deflection, internal force, and stress contour maps of the SCSF at different aspect ratio.

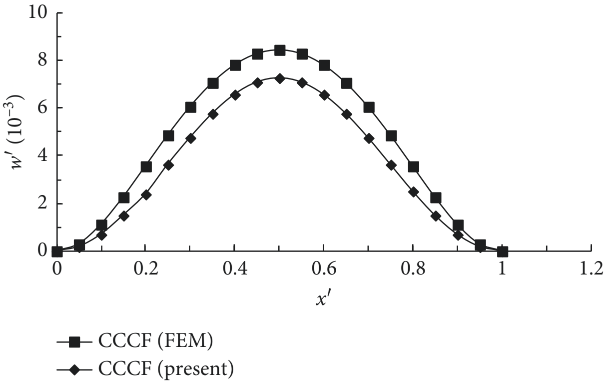

Figure 5: Relationship between dimensionless deflection and x' at $y' = 1.0$ (CCCF).

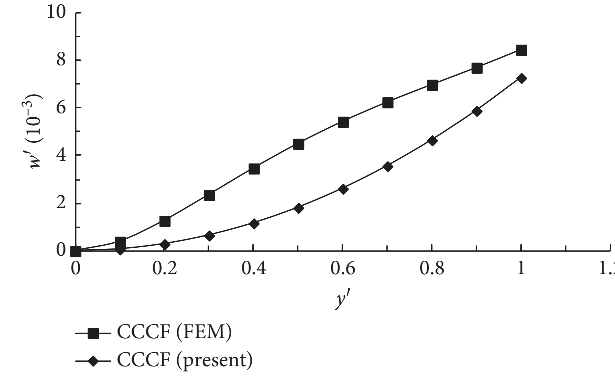

Figure 6: Relationship between dimensionless deflection and y' at $x' = 0.5$ (CCCF).

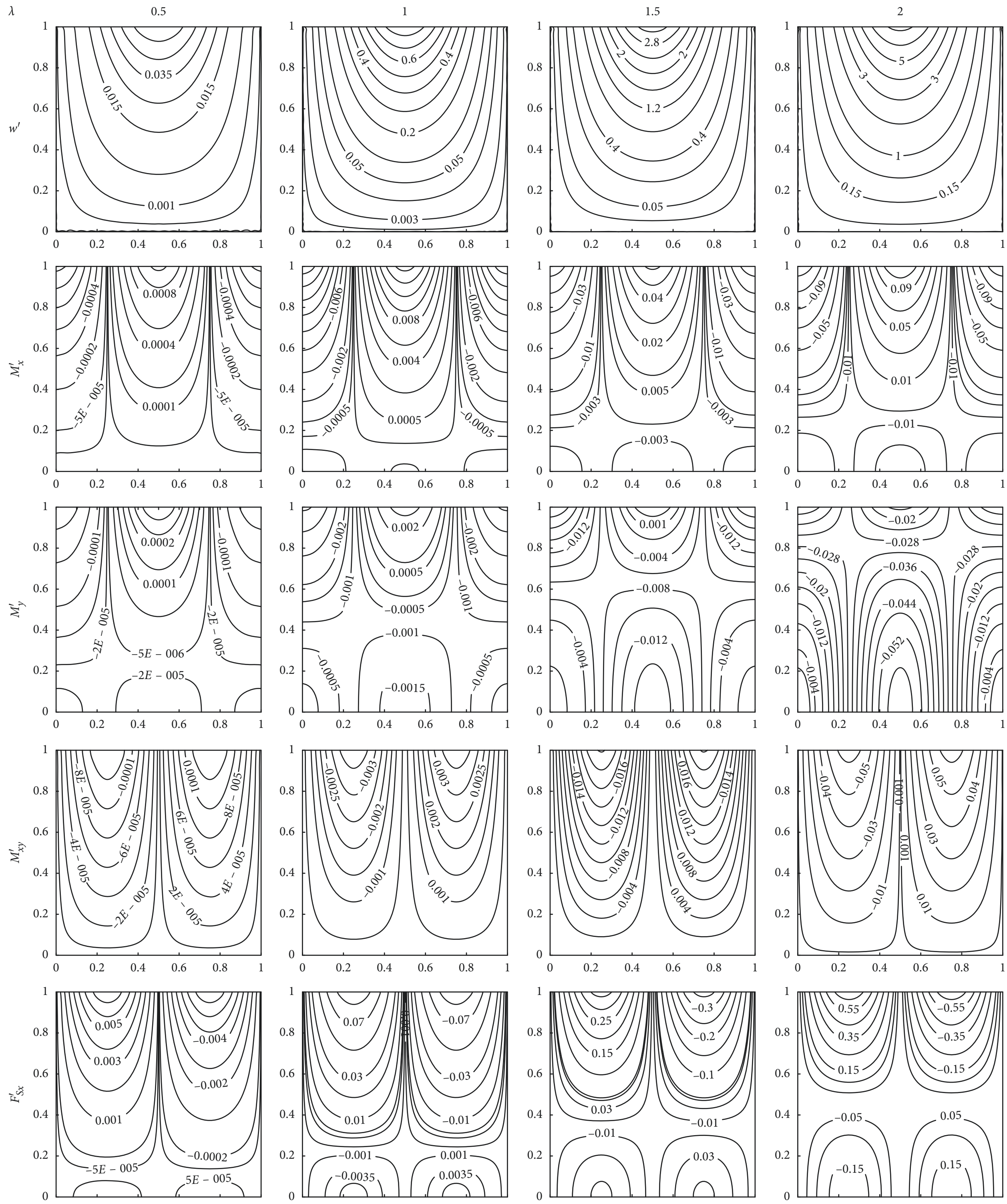

Figure 7: Continued.

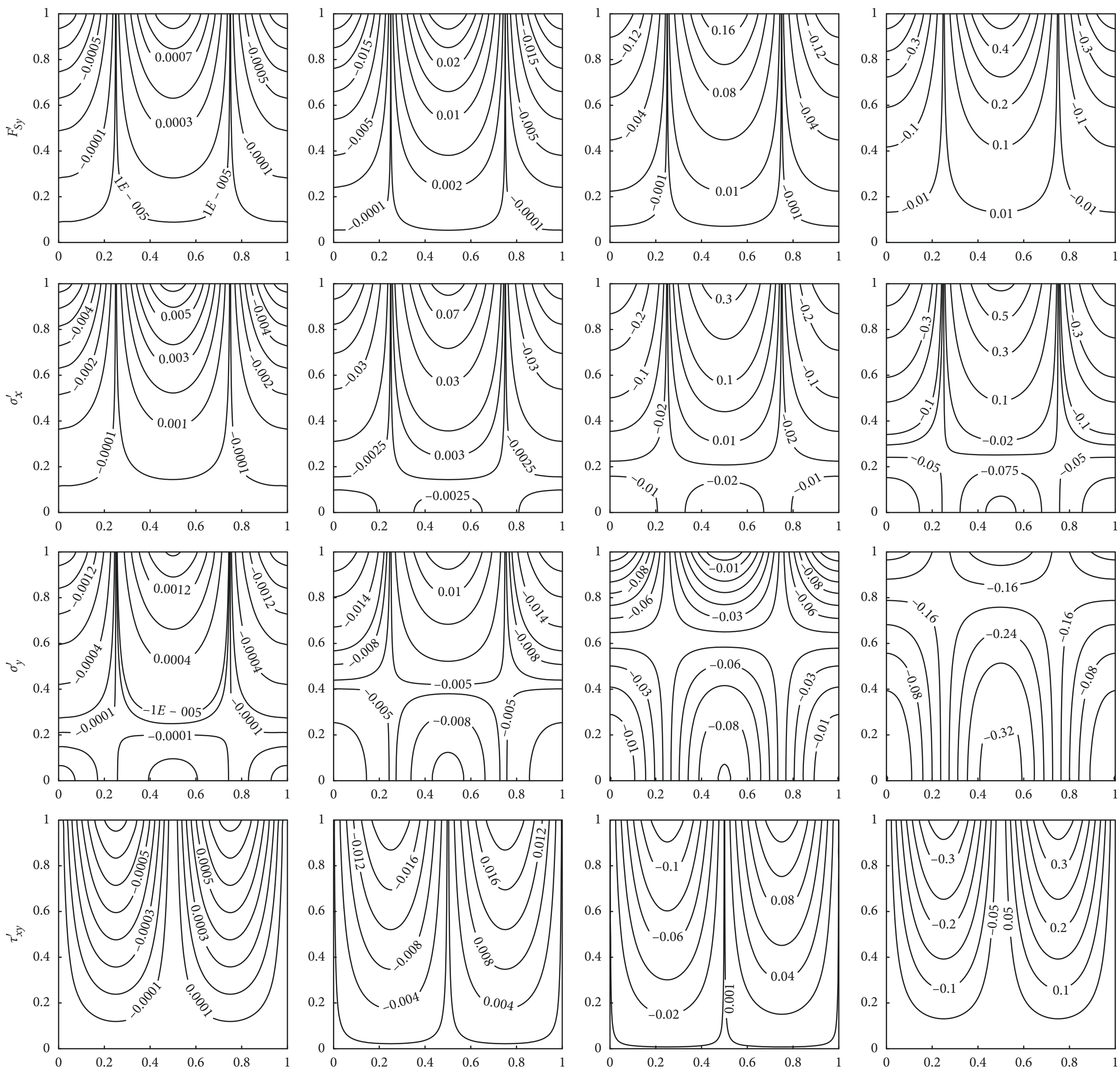

Figure 7: Dimensionless deflection, internal force, and stress contour maps of the CCCF at different aspect ratio.

Table 2: Dimensionless deflection, internal force, and stress of the SCSF rectangular thin plate at $\mu = 0.30$.

$\lambda = a/b$	w' (0.5, 1)/10^{-3}	M'_x (0.5, 1)	M'_y (0.5, 1)	M'_{xy} (1, 1)	F'_{Sx} (0, 1)	F'_{Sy} (0.5, 1)	σ'_x (0.5, 1)	σ'_y (0.5, 1)	τ'_{xy} (1, 1)
0.5	0.2839	−0.0028	0.0007	0.0006	0.0084	0.0028	−0.0171	0.0042	0.0037
1.0	2.7953	−0.0293	0.0027	0.0123	0.0691	0.0552	−0.1756	0.0161	0.0738
1.5	7.4293	−0.0834	−0.0114	0.0490	0.1253	0.2200	−0.5001	−0.0685	0.2941
2.0	12.0074	−0.1473	−0.0605	0.1056	0.0705	0.4740	−0.8840	−0.3630	0.6337

1), (0.5, 1), and (1, 1). The extremum of M'_{xy}, τ'_{xy} is appeared at the points of (0.25, 1) and (0.75, 1). It should be noted that the distribution regularity of the dimensionless internal force and dimensionless stress are different, except the dimensionless deflection. And the dimensionless deflection of CCCF rectangular thin plate is small due to the stronger boundary constraint condition. The maximums of dimensionless deflection, dimensionless internal force, and dimensionless stress of the CCCF rectangular thin plate loaded by the hydrostatic pressure with different values of aspect ratio are shown in Table 3.

4.2. Influence of Poisson's Ratio v on the Deformation and Mechanical Properties of the Rectangular Thin Plates. The maximums of dimensionless deflection, internal force, and

Table 3: Dimensionless deflection, internal force, and stress of the CCCF rectangular thin plate at $\mu = 0.30$.

$\lambda = a/b$	w' (0.5, 1)/10^{-3}	M_x' (0.5, 1)	M_y' (0.5, 1)	M_{xy}' (0.25, 1)	F_{Sx}' (0.75, 1)	F_{Sy}' (0.5, 1)	σ_x' (0.5, 1)	σ_y' (0.5, 1)	τ_{xy}' (0.75, 1)
0.5	0.0637	0.0013	0.0003	−0.0001	−0.0078	0.0013	0.0075	0.0021	0.0008
1.0	0.8735	0.0167	0.0034	−0.0038	−0.1028	0.0345	0.1003	0.0206	0.0231
1.5	3.3566	0.0617	0.0048	−0.0221	−0.3689	0.1988	0.3704	0.0286	0.1329
2.0	7.2531	0.1258	−0.0151	−0.0638	−0.7173	0.5727	0.7546	−0.0904	0.3828

Table 4: Dimensionless deflection, internal force, and stress of the SCSF rectangular thin plate at $\lambda = 2.0$.

v	w' (0.5, 1)/10^{-3}	M_x' (0.5, 1)	M_y' (0.5, 1)	M_{xy}' (1, 1)	F_{Sx}' (0, 1)	F_{Sy}' (0.5, 1)	σ_x' (0.5, 1)	σ_y' (0.5, 1)	τ_{xy}' (1, 1)
0.25	11.3723	−0.1349	−0.0629	0.1072	0.0668	0.4490	−0.8099	−0.3775	0.6431
0.30	12.0074	−0.1473	−0.0605	0.1056	0.0705	0.4740	−0.8840	−0.3630	0.6337
0.35	12.7177	−0.1611	−0.0578	0.1039	0.0747	0.5021	−0.9668	−0.3469	0.6233

Table 5: Dimensionless deflection, internal force, and stress of the CCCF rectangular thin plate at $\lambda = 2.0$.

v	w' (0.5, 1)/10^{-3}	M_x' (0.5, 1)	M_y' (0.5, 1)	M_{xy}' (0.25, 1)	F_{Sx}' (0.75, 1)	F_{Sy}' (0.5, 1)	σ_x' (0.5, 1)	σ_y' (0.5, 1)	τ_{xy}' (0.75, 1)
0.25	6.9544	0.1234	−0.0213	−0.0655	−0.6877	0.5491	0.7402	−0.1279	0.3933
0.30	7.2531	0.1258	−0.0151	−0.0638	−0.7173	0.57527	0.7546	−0.0904	0.3828
0.35	7.5862	0.1284	−0.0083	−0.0619	−0.7495	0.5984	0.7703	−0.0496	0.3714

stress of the SCSF rectangular thin plate and the CCCF rectangular thin plate loaded by the hydrostatic pressure with different values of Poisson's ratio are given in Tables 4 and 5.

As shown in Tables 4 and 5, the dimensionless deflections, internal forces, and stresses of SCSF and CCCF rectangular thin plates loaded by hydrostatic pressure are increased linearly with the values of Poisson's ratio v increased.

5. Conclusions

The deflection, internal force, and stress functions of the SCSF and CCCF rectangular thin plates loaded by the hydrostatic pressure are obtained via the Rayleigh–Ritz method. The method presented in this paper is correct and viable validated by FEM. Moreover, the dimensionless deflection, dimensionless internal force, and dimensionless stress functions of two types of the rectangular thin plates loaded by the hydrostatic pressure are established, which makes this research more general. The dimensionless deflection of the CCCF rectangular thin plate is smaller than the dimensionless deflection of the SCSF rectangular thin plate. The values of dimensionless deflection, dimensionless internal force, and dimensionless stress of the SCSF and CCCF rectangular thin plates are increased with the increasing values of aspect ratio λ and Poisson's ratio μ. The results obtained by this paper can provide the references for the similar thin plates and plane gates in hydraulic engineering.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Additional Points

Highlights. (i) The deflection, internal force, and stress functions of the SCSF and CCCF rectangular thin plates loaded by hydrostatic pressure are established and solved. (ii) The dimensionless deflection, dimensionless internal force, and dimensionless stress functions of the rectangular thin plates with two boundary conditions are obtained. (iii) The influence of aspect ratio λ and Poisson's ratio μ on the deformations and mechanical behaviors of the rectangle thin plate is analyzed.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This research was supported by the National Natural Science Foundation of China (Grant no. 51679199), the Special Funds for Public Industry Research Projects of the Ministry of Water Resources (Grant no. 201501034-04), the Key Laboratory for Science and Technology Co-ordination & Innovation Projects of Shaanxi Province (Grant no. 2014SZS15-Z01), and the Scientific Research Project of Shaanxi Education Department (Grant no. 17JK0573).

References

[1] O. C. Zienkiewicz and Y. K. Cheung, "The finite element method for analysis of elastic isotropic and orthotropic slabs," *Proceedings of the Institution of Civil Engineers*, vol. 28, no. 4, pp. 471–488, 1964.

[2] Z. Shih, "On the spline finite element method," *Journal of Computational Mathematics*, vol. 1, pp. 50–72, 1979.

[3] M. V. Barton, "Finite difference equations for the analysis of thin rectangular plates with combinations of fixed and free edges," Rep. No. 175, Defence Research Laboratory, University of Texas, Austin, TX, USA, 1948.

[4] R. H. MacNeal, "The solution of elastic plate problems by electrical analogies," *Journal of Applied Mechanics*, vol. 18, pp. 59–67, 1951.

[5] W. A. Nash, "Several approximate analyses of the bending of a rectangular cantilever plate by uniform normal pressure," *Journal of Applied Mechanics*, vol. 19, pp. 33–36, 1952.

[6] Y. K. Cheung, *Finite Strip Method in Structural Analysis*, Pergamon Press, Oxford, UK, 1976.

[7] B. M. Donning and W. K. Liu, "Meshless methods for shear-deformable beams and plates," *Computer Methods in Applied Mechanics and Engineering*, vol. 152, no. 1-2, pp. 47–71, 1998.

[8] P. Shen and P. He, "Bending analysis of rectangular moderately thick plates using spline finite element method," *Computers & Structures*, vol. 54, no. 6, pp. 1023–1029, 1995.

[9] A. C. Dixon, "The problem of the rectangular plate," *Journal of the London Mathematical Society*, vol. s1-9, no. 1, pp. 61–74, 1934.

[10] M. K. Huang and H. D. Conway, "Bending of a uniformly loaded rectangular plate with two adjacent edges clamped and the others either simply supported or free," *Journal of Applied Mechanics*, vol. 19, no. 52, pp. 451–460, 1952.

[11] S. P. Timoshenko and S. Woinwsky-Krieger, *Theory of Plates and Shells*, McGraw Hill, New York, NY, USA, 2nd edition, 1959.

[12] J. R. Hutchinson, "On the bending of rectangular plates with two opposite edges simply supported," *Journal of Applied Mechanics*, vol. 59, no. 3, pp. 679–681, 1992.

[13] L. El-Bayoumy, "Application of the extended Kantorovich method to the buckling of rectangular plates of various aspect ratio," *Journal of Engineering Mechanics*, vol. 97, pp. 1277–1289, 1971.

[14] C. E. Imrak and I. Gerdemeli, "The problem of isotropic rectangular plate with four clamped edges," *Sadhana*, vol. 32, no. 3, pp. 181–186, 2007.

[15] F. V. Chang, "Rectangular plates with two adjacent edges clamped and other two adjacent edges free," *Acta Mechanica Solida Sinica*, vol. 4, pp. 491–502, 1981.

[16] C. W. Lim, S. Cui, and W. A. Yao, "On new symplectic elasticity approach for exact bending solutions of rectangular thin plates with two opposite sides simply supported," *International Journal of Solids and Structures*, vol. 44, no. 16, pp. 5396–5411, 2007.

[17] Y. Zhong and R. Li, "Exact bending analysis of fully clamped rectangular thin plates subjected to arbitrary loads by new symplectic approach," *Mechanics Research Communications*, vol. 36, no. 6, pp. 707–714, 2009.

[18] Y. Liu and R. Li, "Accurate bending analysis of rectangular plates with two adjacent edges free and the others clamped or simply supported based on new symplectic approach," *Applied Mathematical Modelling*, vol. 34, no. 4, pp. 856–865, 2010.

[19] R. Li, Y. Zhong, and M. Li, "Analytical bending solutions of free rectangular thin plates resting on elastic foundations by a new symplectic superposition method," *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 469, no. 2153, article 20120681, 2013.

[20] R. Li, B. Wang, G. Li, J. Du, and X. An, "Analytic free vibration solutions of rectangular thin plates point-supported at a corner," *International Journal of Mechanical Sciences*, vol. 96-97, pp. 199–205, 2015.

[21] R. Li, Y. Tian, X. Zheng, H. Wang, S. Xiong, and B. Wang, "New analytic bending solutions of rectangular thin plates with a corner point-supported and its adjacent corner free," *European Journal of Mechanics-A/Solids*, vol. 66, pp. 103–113, 2017.

[22] Y. Khan, P. Tiwari, and R. Ali, "Application of variational methods to a rectangular clamped plate problem," *Computers & Mathematics with Applications*, vol. 63, no. 4, pp. 862–869, 2012.

[23] I. A. Wojtaszak, "The calculation of maximum deflection, moments and shear for uniformly loaded rectangular plate with clamped edge," *Journal of Applied Mechanics*, vol. 4, pp. A173–A176, 1937.

[24] R. L. Taylor and S. Govindjee, "Solution of clamped rectangular plate problems," *Communications in Numerical Methods in Engineering*, vol. 20, no. 10, pp. 757–765, 2004.

[25] D. Young, "Analysis of clamped rectangular plates," *Journal of Applied Mechanics*, vol. 62, pp. A139–A142, 1940.

[26] J. H. Wu, A. Q. Liu, and H. L. Chen, "Exact solutions for free-vibration analysis of rectangular plates using bessel functions," *Journal of Applied Mechanics*, vol. 74, no. 6, pp. 1247–1251, 2007.