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# Multi-Model Reliable Control for Variable Fault Systems under LQG Framework

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**Abstract:** The problem of reliable control for variable fault systems under linear quadratic Gaussian (LQG) framework is studied in this paper. Firstly, a cluster of models is used to cover the dynamic behaviors of different fault modes of a system and, for each model, LQG control is implemented. By using the a posteriori probability of model innovation as the weight information, a multi-model reliable control (MMRC) is proposed. Secondly, it is proved that MMRC can enable the controller to learn the real operating mode of the system. When the controller is in a deadlock state, a deadlock avoidance strategy is given and its convergence of the a posteriori probability is proved. Finally, the validity of MMRC is verified by an example simulation of the lateral-directional control system of an aircraft. Simulation results show that MMRC guarantees an acceptable performance of the closed-loop system. In addition, since the controller fuses the control law of each model according to the weight information, when the system model is switched, the controller implements a soft switching, which avoids the jitter caused by frequent hard switching to the system.

**Keywords:** LQG; variable fault system; a posteriori probability; reliable controller

## 1. Introduction

In the design of LQG controllers, due to the separation property between state estimation and control gain calculation, the design method of control law has received extensive attention from theorists and engineers. To date, a large number of problems in aerospace, aviation, industrial and socio-economic system have achieved satisfactory control effects under LQG framework, and many new methods have been deduced [1–3]. However, all of these methods hold that the system's actuators, sensors, etc. can work properly. When faults occur in the system, these methods not only fail to optimize system performance, but also destabilize the system and even put the system at risk. Therefore, it is thought to be of great theoretical and practical engineering significance to study the design method of the controller under LQG framework in the case of faults appearing in the system.

The LQG framework contains two aspects. On the one hand, the system dynamics are linear, while the disturbance acting on the process and the error affecting the measurements are white Gaussian noise. On the other hand, the performance index is quadratic in state and control with the form of convex function. The existing research results show that a linear state feedback control which makes the performance index of closed-loop system optimal and has the separation property can be determined under the condition that the system is normal. In the general case, the aging of the components, the varying of the environment and other unpredictable factors in the process of control will cause the system to deviate from normal operating conditions, and even cause system faults.

The existing control algorithms are powerless. Faced with this challenge, this paper hopes to design a reliable controller. The reliable controller is thus a controller that can make a closed-loop system with an optimal performance index when the system is in normal conditions and can also make the closed-loop system with an acceptable performance under certain conditions when faults appear in the system. Notice that, when the system is under normal conditions, the controller is the conventional LQG optimal controller. A controller having this property is referred to as a reliable controller in this paper.

The primary role of a reliable controller is to handle system faults. Generally speaking, faults will inevitably occur in a system during its life cycle, such as the measurement deviation of sensor, the actuator stuck, etc. When these faults are reflected in the system model, they cannot be mathematically described by model parameters change, but by the model parameters jump in the high-dimensional discrete space. Essentially, each point in the discrete space corresponds to a model, and different control strategies are designed based on different models [4–6]. If the fault model corresponding to any point in the discrete space is known, the number of models will have a dimension disaster, and it is impossible to refine the faults to different degrees to obtain all the models. A viable strategy is to build a cluster of models that traverse all possible faults as much as possible. The research results in this paper show that the reliable controller can be designed for a limited system model set. The controller tends to LQG optimal feedback control over time when the system is normal, and it can make the system have acceptable performance when the faults occur in the system.

A prerequisite for the reliable controller design is to approximate the various conditions of the system with a cluster of models which includes a normal model with no faults and several known fault models. The a priori information of the fault model is based on an understanding and mastery of history of the system, especially for systems that operate repeatedly in different cycles, such as aeronautical vehicles. There have been some methods for the problem of controller design with a model set that has multi-model control strategies [7,8]. These methods require that the controller is able to detect the fault model immediately. As system faults differ, the control laws matching different models are constantly switched according to the switching indicators, which often leads to strong jitter in the system at the switching point [9,10]. This is a hard switching method. Although it can make the system response faster, the jitter is unavoidable. In fact, a variety of methods that are accompanied by the development of sliding mode variable structure technology appear to solve the jitter.

In order to avoid the jitter, soft switching methods came into being [11,12]. In this paper, a soft switching among multiple models is realized by means of the dual adaptive control method. Since Feldbaum first proposed the dual control method for the autoregressive moving average (ARMA) model with unknown parameters, after more than half a century of research, it has been proved that dual control can optimize the system toward the desired target on one hand, and on the other hand, it can actively collect information and guarantee the unknown parameter estimation process [13–15].

Based on the above analysis, the contribution in this paper is as follows: (1) It is assumed in this paper that the normal model and the possible fault models are known. LQG optimal control is applied for each model. By using the a posteriori probability of each model as the weight information, an MMRC is proposed; (2) It is the optimal LQG controller when the system is normal, and the controller performs reliably when faults occur in the system; (3) It neither needs to detect the fault model, nor needs to detect the fault time; (4) It implements a soft switching among the multiple models, which avoids the jitter caused by hard switching.

The remainder of this paper is organized as follows: the problem to be solved in this paper will be presented in Section 2. The theoretical basis of MMRC will be established in Section 3. Section 4 illustrates the validity of the control algorithm through an example of the lateral-directional control system of an aircraft, and, finally, the conclusions will be offered in Section 5.

## 2. Problem Statement

Consider the following discrete-time stochastic linear systems:

$$x(k+1) = [G(k) + \Delta G_j] x(k) + [H(k) + \Delta H_j] u(k) + \omega(k), \quad (1)$$

$$z(k) = [C(k) + \Delta C_j] x(k) + v(k), \quad (2)$$

where  $x(k)$  is an  $n$ -dimensional vector of state,  $z(k)$  is a  $p$ -dimensional vector of output and  $u(k)$  is an  $m$ -dimensional vector of control input. The process noise  $\omega(k)$ , observation noise  $v(k)$  and initial condition  $x(0)$  are mutually independent white Gaussian, whose statistics are  $\omega(k) \sim \mathcal{N}(0, W)$ ,  $v(k) \sim \mathcal{N}(0, V)$ , and  $x(0) \sim \mathcal{N}(0, P_0)$ .  $G(k)$ ,  $H(k)$  and  $C(k)$  are matrices of appropriate dimensions, and their variable quantities  $\{\Delta G_j, \Delta H_j, \Delta C_j\}$  represent the deviation of system components, actuators and sensors when the system is in operating mode  $j$ ,  $j \in \Omega = \{0, 1, \dots, s\}$ . When  $j = 0$ , it represents that the system is in normal mode and there is no fault, and the corresponding  $\Delta G_0 = \Delta H_0 = \Delta C_0 = 0$ . When  $j = 1, 2, \dots, s$ , it represents that the system is in different fault modes respectively, and, for any fault mode  $\{\Delta G_j, \Delta H_j, \Delta C_j\}$  corresponds to a set of determined values. In this paper, it is assumed that  $\{\Delta G_j, \Delta H_j, \Delta C_j\}$  are known a priori; that is, the model parameters corresponding to the mode  $j$  are known, but which mode the system is in is unknown. In addition, the system can only be in one mode during the same stage, and, in different stages, the system mode is variable.

The performance index for the system is quadratic in the state and control

$$J = x^T(N) A x(N) + \sum_{k=0}^{N-1} [x^T(k) A x(k) + u^T(k) B u(k)], \quad (3)$$

where  $A$  and  $B$  are positive semi-definite and positive definite symmetric matrices of appropriate dimensions, respectively.

Define the information set at time  $k$  to be  $I^k$

$$I^k = \{u(0), \dots, u(k-1); z(1), \dots, z(k)\}.$$

The problem to be solved in this paper is to find a feedback control law  $u(k) = f_k(I^k)$  such that the expected performance index  $E\{J\}$  of system (1) and (2) is minimized, namely,

$$\begin{aligned} (P) \quad & \min E\{J\} \\ \text{s.t.} \quad & x(k+1) = [G(k) + \Delta G_j] x(k) + [H(k) + \Delta H_j] u(k) + \omega(k), \\ & z(k) = [C(k) + \Delta C_j] x(k) + v(k), j \in \Omega. \end{aligned}$$

Note that, if the process model parameters are completely known, the above control problem becomes the classical LQG control problem, for which there is already a mature solution. However, the problem (P) solved in this paper is different from LQG, namely, the fault at current time is unknown and which fault is also unknown.

## 3. Reliable Controller Design

For the convenience of problem statement and notation, systems (1) and (2) can be rewritten as follows:

$$x_j(k+1) = G_j x_j(k) + H_j u(k) + \omega(k), \quad (4)$$

$$z(k) = C_j x_j(k) + v(k), \quad (5)$$

where  $G_j = G(k) + \Delta G_j$ ,  $H_j = H(k) + \Delta H_j$ ,  $C_j = C(k) + \Delta C_j$ ,  $x_j(k)$  is the state of the system under mode  $j$ , and  $z(k)$  is the output.

**Definition 1.** When the system is in mode  $j$ , the state estimate based on real-time information set  $I^k$  is

$$\hat{x}_j(k) = E \left\{ x_j(k) \mid I^k \right\},$$

$\hat{x}_j(k)$  can be obtained by the Kalman filter, [16]

$$\hat{x}_j(k) = G_j \hat{x}_j(k-1) + H_j u(k-1) + K_j(k) \tilde{z}_j(k), \tag{6}$$

where

$$K_j(k) = P_j(k|k-1) C_j^T \left[ C_j P_j(k|k-1) C_j^T + V \right]^{-1}, \tag{7}$$

$$P_j(k|k-1) = G_j P_j(k-1) G_j^T + W, \tag{8}$$

$$P_j(k) = [I - K_j(k) C_j] P_j(k|k-1), \tag{9}$$

$$\tilde{z}_j(k) = z(k) - C_j [G_j \hat{x}_j(k-1) + H_j u(k-1)], \tag{10}$$

with initial condition  $\hat{x}_j(0) = \hat{x}(0)$  and  $P_j(0) = P_0$ .  $K_j(k)$  being the filter gain.

**Theorem 1.** When the system is in mode  $j$ , the control law that minimizes performance index  $E \{J\}$  is given by

$$u_j^*(k) = -L_j(k) \hat{x}_j(k), \tag{11}$$

where, for  $k = N-1, N-2, \dots, 0$ ,

$$L_j(k) = D_j^{-1}(k) H_j^T S_j(k+1) G_j, \tag{12}$$

$$D_j(k) = H_j^T S_j(k+1) H_j + B, \tag{13}$$

$$S_j(k) = G_j^T S_j(k+1) G_j + A - L_j^T(k) D_j(k) L_j(k), \tag{14}$$

with the boundary condition  $S_j(N) = A$ .  $\hat{x}_j(k)$  can be obtained from Definition 1.

**Proof of Theorem 1.** Let the optimal cost-to-go of the operating mode  $j$  at time  $k$  be  $J_j^*(k)$

$$J_j^*(k) = \min_{u_j(k), u_j(k+1), \dots, u_j(N-1)} E \left\{ x_j^T(N) A x_j(N) + \sum_{i=k}^{N-1} \left[ x_j^T(i) A x_j(i) + u_j^T(i) B u_j(i) \right] \mid I^k \right\}.$$

According to the smoothing property of expectations and optimal theory of stochastic dynamic programming, we have

$$J_j^*(k) = \min_{u_j(k)} E \left\{ x_j^T(k) A x_j(k) + u_j^T(k) B u_j(k) + J_j^*(k+1) \mid I^k \right\}, \tag{15}$$

with the boundary condition

$$J_j^*(N) = E \left\{ x_j^T(N) A x_j(N) \mid I^N \right\}. \tag{16}$$

The solution to (P) can be obtained by using dynamic programming. Let  $l$  denote the time. If (11) is true at the initial time  $l = N-1$ , let it be true when  $l = k+1$ . If it can be proved that it is true when  $l = k$ , then the theorem is proved.

When  $l = N-1$ , we have

$$J_j(N-1) = E \left\{ x_j^T(N-1) A x_j(N-1) + u_j^T(N-1) B u_j(N-1) + J_j^*(N) \mid I^{N-1} \right\}. \tag{17}$$

Substituting (16) into (17), according to the nature of trace and Definition 1, yields

$$J_j(N-1) = \text{tr}(AW) + \hat{x}_j^T(N-1) \left( G_j^T A G_j + A \right) \hat{x}_j(N-1) + 2u_j^T(N-1) H_j^T A G_j \hat{x}_j(N-1) + u_j^T(N-1) \left( H_j^T A H_j + B \right) u_j(N-1) + \text{tr} \left( \left( G_j^T A G_j + A \right) P_j(N-1) \right). \tag{18}$$

Equation (18) is quadratic in control  $u_j(N-1)$ . Letting  $\frac{dJ_j(N-1)}{du_j(N-1)} = 0$ , we can get the LQG optimal control  $u_j^*(N-1)$  at time  $N-1$ ,

$$u_j^*(N-1) = -L_j(N-1) \hat{x}_j(N-1), \tag{19}$$

where  $L_j(N-1)$  is identical to (12). Substituting (19) into (18), then the optimal cost-to-go at the time  $N-1$  is given as

$$J_j^*(N-1) = \hat{x}_j^T(N-1) S_j(N-1) \hat{x}_j(N-1) + s_j(N-1),$$

where  $S(N-1)$  is identical to (11), and  $s(N-1)$  is uncorrelated to control and state variables

$$s_j(N-1) = \text{tr} \left[ A \left( G_j P_j(N-1) G_j^T + W \right) \right].$$

Assume that at time  $l = k + 1$  we have

$$J_j^*(k+1) = \hat{x}_j^T(k+1) S_j(k+1) \hat{x}_j(k+1) + s_j(k+1). \tag{20}$$

According to (15), we can get

$$J_j^*(k) = \min_{u_j(k)} E \left\{ \hat{x}_j^T(k) A x_j(k) + u_j^T(k) B u_j(k) + J_j^*(k+1) \mid I^k \right\}. \tag{21}$$

Substituting (20) into (21) by mathematical induction yields the following:

$$J_j(k) = \hat{x}_j^T(k) \left( G_j^T S_j(k+1) G_j + A \right) \hat{x}_j(k) + 2u_j^T(k) H_j^T S_j(k+1) G_j \hat{x}_j(k) + u_j^T(k) \left( H_j^T S_j(k+1) H_j + B \right) u_j(k) + \text{tr} \left( \left( G_j^T S_j(k+1) G_j + A \right) P_j(k) + AW \right).$$

Similarly, letting  $\frac{dJ_j(k)}{du_j(k)} = 0$ , we get the LQG optimal control

$$u_j^*(k) = -L_j(k) \hat{x}_j(k),$$

where  $L_j(k)$  and (12) are identical. Substituting  $u_j^*(k)$  into performance index  $J_j(k)$ , we get

$$J_j^*(k) = \hat{x}_j^T(k) S_j(k) \hat{x}_j(k) + s_j(k),$$

where  $S_j(k)$  is identical to (14), and  $s_j(k)$  is uncorrelated to control and state variables

$$s_j(k) = s_j(k+1) + \text{tr} \left[ A \left( G_j P_j(k) G_j^T + W \right) \right].$$

This completes the proof.  $\square$

During the operation, the system may be in the normal mode, or switch back and forth among  $s$  fault modes. According to Theorem 1, each mode corresponds to an LQG optimal control. How to apply reliable control to the system based on  $s + 1$  LQG optimal control, the following theorem answers this question.

**Theorem 2.** The control law that minimizes the performance index  $E \{J\}$  of the system (1) and (2) at time  $k$  is

$$u^*(k) = \sum_{j=0}^s q_j(k) u_j^*(k), \tag{22}$$

where  $u_j^*(k)$  is the LQG optimal control law when the system is in mode  $j$ ,  $q_j(k)$  is the a posteriori probability of mode  $j$  and satisfies the following recursive equation:

$$q_j(k) = \frac{M_j(k)}{\sum_{i=0}^s M_i(k) q_i(k-1)} q_j(k-1), \tag{23}$$

where

$$M_j(k) = |\Delta_j(k)|^{-\frac{1}{2}} \exp \left[ -\frac{1}{2} \tilde{z}_j^T(k) \Delta_j^{-1}(k) \tilde{z}_j(k) \right], \tag{24}$$

$$\Delta_j(k) = C_j P_j(k|k-1) C_j^T + V, \tag{25}$$

with the boundary condition  $q_j(0) = 1/(s+1)$ .  $P_j(k|k-1)$  and  $\tilde{z}_j$  are identical to (8) and (10), respectively.

**Proof of Theorem 2.** According to the stability theorem of filter, it can be obtained that, when  $k \rightarrow \infty$ , the limit of  $\Delta_j(k)$  is a constant matrix that is denoted as  $\Delta_j$ . Let  $i$  be the real system, for the non-real system  $j, j \neq i$ , define  $\pi_j(k) = \frac{q_j(k)}{q_i(k)}$ , according to (23), we have

$$\pi_j(k) = \frac{q_j(k)}{q_i(k)} = \frac{M_j(k)}{M_i(k)} \pi_j(k-1) = \pi_j(0) \prod_{n=1}^k \frac{M_j(n)}{M_i(n)}. \tag{26}$$

Substituting (24), (25) and the boundary condition  $q_j(0) = 1/(s+1)$  into (26)

$$\pi_j(k) = \left( \prod_{n=1}^k \left| \frac{\Delta_i}{\Delta_j} \right|^{\frac{1}{2}} \right) \exp \left\{ -\frac{1}{2} \sum_{n=1}^k \left[ \tilde{z}_j^T(n) \Delta_j^{-1} \tilde{z}_j(n) - \tilde{z}_i^T(n) \Delta_i^{-1} \tilde{z}_i(n) \right] \right\}, \tag{27}$$

where  $\Delta_j(k)$  is replaced by its limit  $\Delta_j$ . Taking the logarithm on both sides of (27), and, according to the nature of trace, we get

$$\ln \pi_j(k) = \frac{n}{2} \ln \left| \frac{\Delta_i}{\Delta_j} \right| - \frac{1}{2} \text{tr} \left\{ \sum_{n=1}^k \tilde{z}_j(n) \tilde{z}_j^T(n) \Delta_j^{-1} \right\} + \frac{1}{2} \text{tr} \left\{ \sum_{n=1}^k \tilde{z}_i(n) \tilde{z}_i^T(n) \Delta_i^{-1} \right\}. \tag{28}$$

Since  $i$  is a real system, we have

$$E \left\{ \tilde{z}_i(k) \tilde{z}_i^T(k) \right\} = \Delta_i(k).$$

According to the ergodicity of the innovation sequence  $\tilde{z}_i(k)$ , it can be obtained that, when  $k \rightarrow \infty$ ,

$$\frac{1}{k} \sum_{n=1}^k \tilde{z}_i(n) \tilde{z}_i^T(n) \Delta_i^{-1} \rightarrow I. \tag{29}$$

For the non-real system  $j$ , innovation

$$\tilde{z}_j(k) = z(k) - C_j \hat{x}_j(k|k-1) = C_i x_i(k) + v_i(k) - C_i \hat{x}_i(k|k-1) + C_i \hat{x}_i(k|k-1) - C_j \hat{x}_j(k|k-1), \tag{30}$$

let

$$e_i(k|k-1) = \hat{x}_i(k|k-1) - x_i(k), \tag{31}$$

$$F_j(k) = C_i \hat{x}_i(k|k-1) - C_j \hat{x}_j(k|k-1). \tag{32}$$

Substituting (31) and (32) into (30) yields

$$\tilde{z}_j(k) = v_i(k) - C_i e_i(k|k-1) + F_j(k).$$

Since  $v_i(k)$  and  $e_i(k|k-1)$  are uncorrelated, according to (25), we get

$$\begin{aligned} E \left\{ \tilde{z}_j(k) \tilde{z}_j^T(k) \right\} &= E \left\{ C_i e_i(k|k-1) e_i^T(k|k-1) C_i^T + v_i(k) v_i^T(k) + F_j(k) F_j^T(k) \right\} \\ &= C_i P_i(k-1|k) C_i^T + V + E \left\{ F_j(k) F_j^T(k) \right\} \\ &= \Delta_i(k) + E \left\{ F_j(k) F_j^T(k) \right\}. \end{aligned}$$

Notice that  $F_j(k) F_j^T(k)$  is a semi-positive matrix such that  $E \left\{ \tilde{z}_j(k) \tilde{z}_j^T(k) \right\} \geq \Delta_i(k)$  holds if and only if  $j = i$ ; then, we have

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{n=1}^k \tilde{z}_j(n) \tilde{z}_j^T(n) \geq \Delta_i. \tag{33}$$

Substituting (29), (33) into (28), when  $k \rightarrow \infty$ , we get

$$\frac{2}{k} \ln \pi_j(k) \rightarrow \ln \left| \frac{\Delta_i}{\Delta_j} \right| - \text{tr} \left( \Delta_j^{-1} \Delta_i \right) + \text{tr}(I) - \text{tr} \left( \Delta_j^{-1} \Sigma_j \right), \tag{34}$$

where

$$\Sigma_j = \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{n=1}^k \tilde{z}_j(n) \tilde{z}_j^T(n) - \Delta_i.$$

Since the right side of (34) is negative, when  $k \rightarrow \infty$

$$\frac{2}{k} \ln \pi_j(k) \rightarrow -c,$$

where  $c > 0$ . Then, we have

$$\pi_j(k) = K \exp \left( -\frac{kc}{2} \right), \tag{35}$$

where  $K$  is a constant. Therefore, when  $k \rightarrow \infty$ , we have  $\frac{q_j(k)}{q_i(k)} \rightarrow 0$ , so we get  $q_j(k) \rightarrow 0$ , since  $\sum_{j=0}^s q_j(k) = 1$ , we get  $q_i(k) \rightarrow 1$ .

At this time, the reliable control law  $u^*(k) = \sum_{j=0}^s q_j(k) u_j^*(k)$  applied to the real system will tend to  $u_i^*(k)$ . This completes the proof.  $\square$

The above assumption of the boundary condition  $q_j(0) = 1/(s+1)$  indicates that the reliable controller does not have any preference for the system modes at the initial time, which is the worst case from the viewpoint of probability, even though the reliable controller will still tend to the control law of the real system. Theorem 2 demonstrates the reliability of the controller, and also shows that MMRC implements a soft switching strategy.

Note that, when  $k \rightarrow \infty$ ,  $q_i(k)$ , which is the a posteriori probability of true model  $i$ , tends to 1, while  $q_j(k)$  ( $j \in \Omega, j \neq i$ ) tends to 0. Suppose at some time unit  $k$ ,  $q_i(k) = 1, q_j(k) = 0$ ; it can be seen from (23) that  $q_i(k)$  and  $q_j(k)$  will no longer change. According to (22), it yields  $u^*(k) \equiv u_i^*(k)$ , namely,  $u^*(k)$  always equals the control law of model  $i$ . If the true model changes from  $i$  to  $j$  in the next stage,  $u^*(k)$  will not change to  $u_j(k)$ , which is referred to as “lock out”. The following theorem answers the question of how to unlock a posteriori probability.

**Theorem 3.** Assume that if the a posteriori probability of the system is locked out at time  $k$ , that is, the a posteriori probability of real system  $i$ ,  $q_i(k) = 1$ , and the a posteriori probabilities of non-real system  $j$ ,  $q_j(k) = 0, j \neq i$ , then, through the following transformation,

$$\begin{aligned} q_i(k) &= 1 - \varepsilon, \\ q_j(k) &= \varepsilon/s, j \neq i, \end{aligned} \tag{36}$$

where  $0 < \varepsilon \ll 1$ , the a posteriori probability of the system after time  $k$  not only can be unlocked, but also does not affect the convergence.

**Proof of Theorem 3.** According to (23), we have

$$q_i(k+m) = \frac{M_i(k+m)q_i(k+m-1)}{\sum_{j=0}^s M_j(k+m)q_j(k+m-1)} = \frac{1}{1 + \sum_{j=0, j \neq i}^s \frac{M_j(k+m)q_j(k+m-1)}{M_i(k+m)q_i(k+m-1)}} = \frac{1}{1 + \frac{q_j(k)}{q_i(k)} \sum_{j=0, j \neq i}^s \prod_{n=1}^m \frac{M_j(k+n)}{M_i(k+n)}}. \tag{37}$$

Substituting (36) into (37), we get

$$q_i(k+m) = \frac{1}{1 + \frac{\varepsilon}{s(1-\varepsilon)} \sum_{j=0, j \neq i}^s \prod_{n=1}^m \frac{M_j(k+n)}{M_i(k+n)'}}$$

where  $\varepsilon/s(1-\varepsilon)$  is a constant. Let  $\xi_j(k+m) = \prod_{n=1}^m \frac{M_j(k+n)}{M_i(k+n)}$ , according to (24) and (25), we have

$$\begin{aligned} \xi_j(k+m) &= \exp \left\{ -\frac{1}{2} \sum_{n=1}^m \left[ \bar{z}_j^T(k+n) \Delta_j^{-1}(k+n) \bar{z}_j(k+n) - \bar{z}_i^T(k+n) \Delta_i^{-1}(k+n) \bar{z}_i(k+n) \right] \right\} \times \\ &\quad \left( \prod_{n=1}^m \left| \frac{\Delta_j(k+n)}{\Delta_i(k+n)} \right|^{\frac{1}{2}} \right). \end{aligned} \tag{38}$$

Comparing (38) with (27), similarly, we can prove that, when  $m \rightarrow \infty$ , if  $i$  is still real system, we have  $\xi_j(k+m) \rightarrow 0$  and then we can get  $q_i(k+m) \rightarrow 1$ ; if  $i$  becomes non-real system, then  $\xi_j(k+m) \rightarrow \infty$  and  $q_i(k+m) \rightarrow 0$ . This completes the proof. □

One can see from Theorem 3 that  $\varepsilon$  is a constant, whose value only affects the convergence speed of the a posteriori probability and does not affect the convergence. That is, after the time  $k$ , no matter whether the system mode changes, the a posteriori probability of the real system will always tend to 1, and a posteriori probability of the non-real system will tend to 0.

#### 4. Simulation Analysis

In summary, the design of the reliable  $\Delta$  controller can be implemented by the following algorithm:

- Step 1: Calculate the Kalman gain  $K_j(k)$  offline according to (7)–(9).
- Step 2: Calculate control gain  $L_j(k)$  offline according to (12)–(14).
- Step 3: Set  $k = 0$ .
- Step 4: Calculate  $\bar{z}_j(k)$  and  $\hat{x}_j(k|k)$  according to (10) and (6), respectively.

- Step 5: Calculate  $u_j^*(k)$  according to (11).
- Step 6: Calculate  $q_j(k)$  according to (23).
- Step 7: Calculate control law  $u^*(k)$  according to (22), and apply this control to the system.
- Step 8: Update  $q_j(k)$  according to (36).
- Step 9: If  $k = N - 1$ , stop. Otherwise, set  $k := k + 1$ , go back to Step 4.

In order to illustrate the characteristics of the controller designed in this paper, an example of an aircraft lateral-directional control system [17] is given as follows:

$$\begin{aligned} x(k+1) &= G_j x(k) + H_j u(k) + \omega(k), \\ z(k) &= x(k) + v(k), \end{aligned}$$

where system state  $x = [\varphi \ \beta \ p_s \ r_s]^T$  and control input  $u = [\delta_a \ \delta_r]^T$ .  $\varphi, \beta, p_s$  and  $r_s$  represent the bank angle, the sideslip angle, the body roll rate, and the body yaw rate, respectively.  $\delta_a$  and  $\delta_r$  denote the differential aileron and the rudder, which control the roll and yaw motion, respectively. The statistics for the process noise, observation noise and initial condition are  $\omega(k) \sim \mathcal{N}(0, 0.01I_4)$ ,  $v(k) \sim \mathcal{N}(0, 0.01I_4)$  and  $x(0) \sim \mathcal{N}(0, I_4)$  respectively where  $I_4$  is an identity matrix of  $4 \times 4$  dimension.

The performance index is

$$J = E \left\{ x^T(50) x(50) + \sum_{k=0}^{49} [x^T(k) x(k) + u^T(k) u(k)] \right\}.$$

Assume that the airspeed is excessive at fault time  $k_f = 25$  due to the abnormality of the pitot tube system, and the process model parameters related to pitot tube are attenuated to 90% or 80%. That is, there are three potential operating modes in the system, i.e.,  $j \in \Omega = \{0, 1, 2\}$ , and the operation process of system is divided into two stages by  $k_f$ . Before  $k_f$ , we call it the first stage, and, after  $k_f$  we call it the second stage. It is further assumed that the real system is  $j = 0$  in the first stage and in the second stage the real system is  $j = 2$ . When the system is normal, i.e.,  $j = 0$ , its model parameters are given as follows:

$$G_0 = \begin{bmatrix} 1 & 0 & 0.1 & 0 \\ 0.0049 & 0.9917 & 0 & -0.1 \\ 0 & -0.4545 & 0.8331 & 0.0172 \\ 0 & 0.3882 & -0.0065 & 0.9911 \end{bmatrix}, H_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0.0001 \\ 0.2728 & 0.0058 \\ 0.0040 & -0.0136 \end{bmatrix}.$$

When the faults appear in the system, the relevant model parameters are 90% of the normal system, i.e.,  $j = 1$ ,

$$\begin{aligned} G_1(2,1) &= G_0(2,1) \times 90\%, \\ G_1(2,2) &= G_0(2,2) \times 90\%, \\ H_1(2,2) &= H_0(2,2) \times 90\%, \end{aligned}$$

and 80% of the normal system, i.e.,  $j = 2$ ,

$$\begin{aligned} G_2(2,1) &= G_0(2,1) \times 80\%, \\ G_2(2,2) &= G_0(2,2) \times 80\%, \\ H_2(2,2) &= H_0(2,2) \times 80\% \end{aligned}$$

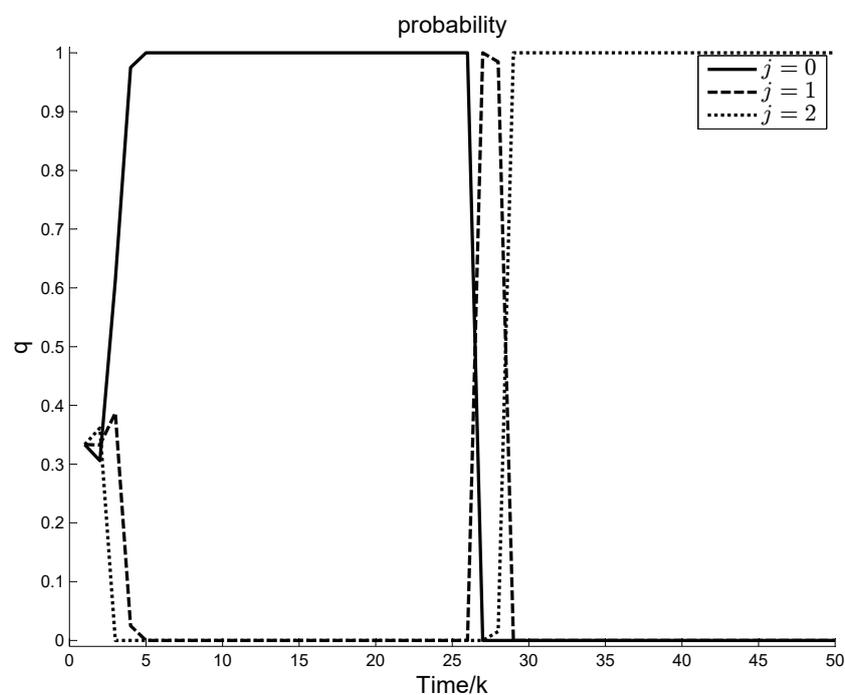
where  $G_j(2,1)$ ,  $G_j(2,2)$  and  $H_j(2,1)$  represent the elements of the second row and the first column, the second row and the second column in the matrix  $G$ , and the second row and second column in

the matrix  $H$  respectively when the system is in mode  $j$ . Note that we do not know which mode the system is in, and we do not know if the system mode is switched. We only know no matter which mode the system is in, its corresponding model is covered by the known model set  $\Omega$ .

In this paper, two control strategies are used to implement example simulation. One is MMRC under the above conditions, the other is optimal control (OC) under the condition that the process model parameters are completely known. The performance index corresponding to OC should be the lower bound of the other controls. Simulation results are shown in Figures 1–5. Figure 1 illustrates the variation curve of the a posteriori probabilities of different modes in  $\Omega$  under MMRC. Figures 2 and 3 show the system control components under OC and MMRC, respectively. Figure 4 indicates the response curve of a representative system state. Performance index of the two control strategies are demonstrated in Figure 5.

It is observed in Figure 1 that in the above two stages the a posteriori probabilities of different system modes both have experienced change and stability. In the first stage, it is because the system is assumed with equal probability of  $1/3$  at the initial time, and, in the second stage, it is because a fault occurs in the system. As the system continuously obtains measurement results, the a posteriori probability of  $j = 0$  tends to 1 and the a posteriori probabilities of  $j = 1, j = 2$  tend to 0 in the first stage, and in the second stage the a posteriori probability of  $j = 2$  tends to 1 and the a posteriori probabilities of  $j = 0, j = 1$  tend to 0. It is noted that the change of the a posteriori probability of system is not 0 or 1, but varies between 0 and 1, which indicates that MMRC implements soft switching.

Figures 2 and 3 demonstrate that, in the first stage, MMRC follows closely OC except for the initial transient period. When fault occurs in the system in the second stage, a certain error lasts for a while, but then MMRC fluctuates around OC and eventually MMRC follows closely OC. It is observed in Figure 4 that, in the first stage, there is only a slight difference in the state response curves under OC and MMRC, and, in the second stage, there is a deviation. However, the state response curve of MMRC finally follows closely that of OC in the control horizon.



**Figure 1.** The *a-posteriori* probabilities of different modes of the system under MMRC.

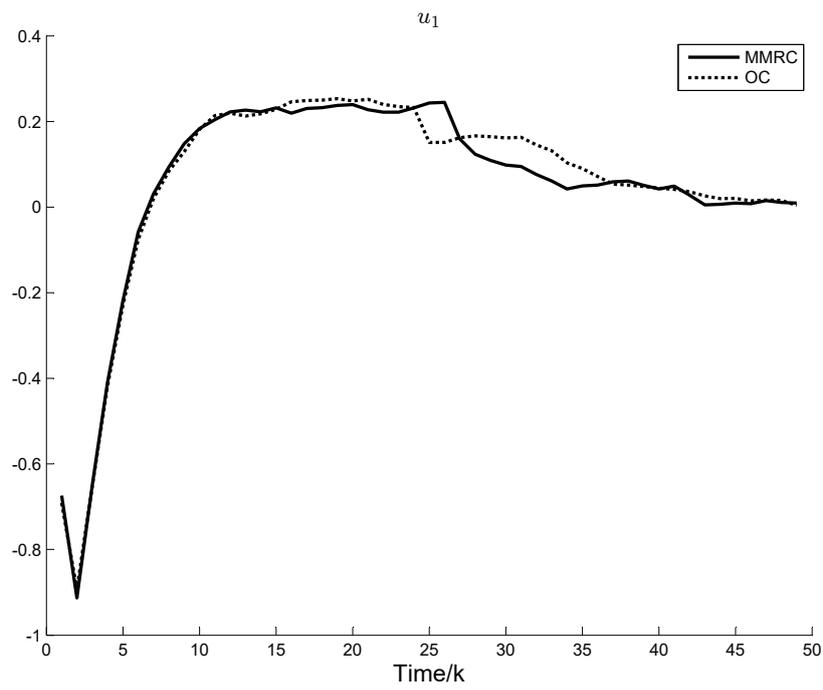


Figure 2. Control component  $u_1$ .

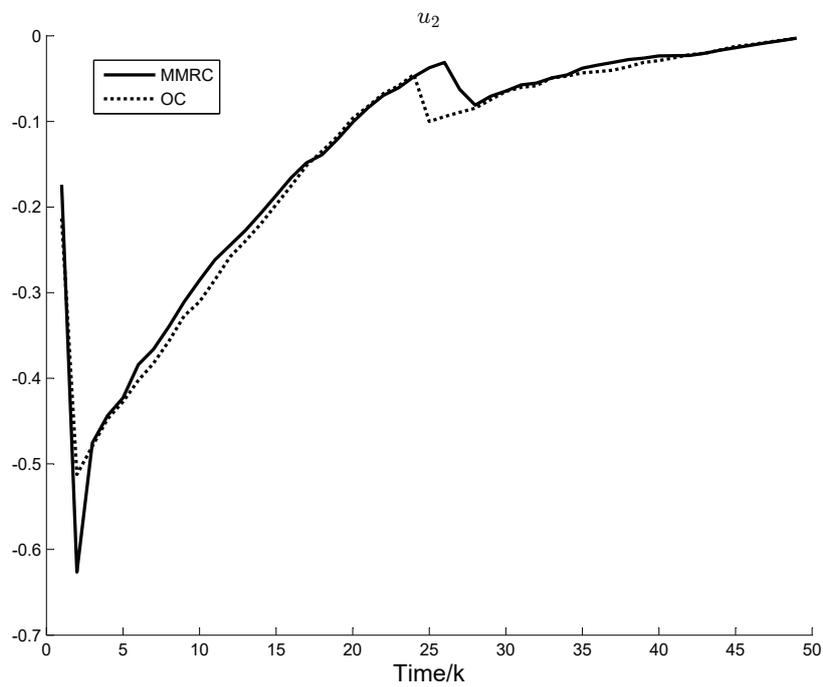


Figure 3. Control component  $u_2$ .

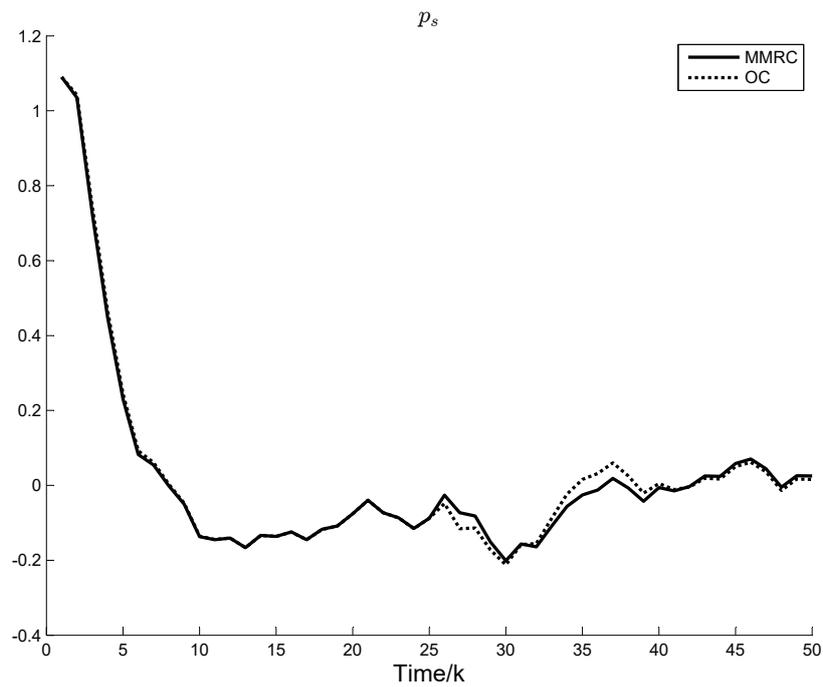


Figure 4.  $p_s$  under MMRC and OC.

Figure 5 demonstrates that only in the second stage is there a certain deviation between the performance index of MMRC and OC. These phenomena are due to two aspects. One is the switching of control law between multiple models, the other is the pursuit of control objective. Notice that it is an inevitable cost for MMRC.

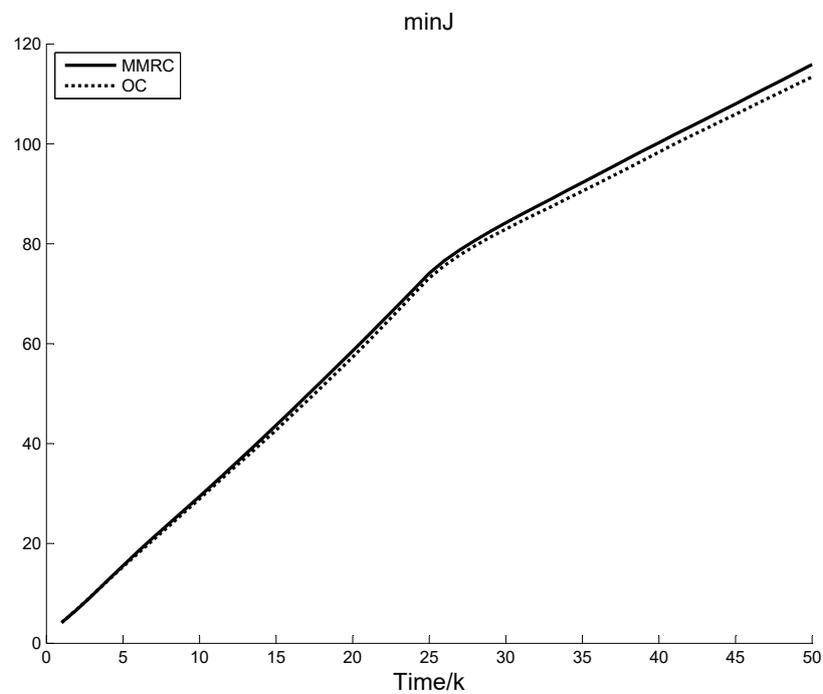


Figure 5. The performance index under MMRC and OC.

## 5. Conclusions

In this paper, an MMRC, which is under the condition that the models are known, is given for controlling the variable fault system under LQG framework. The controller fuses the control law of each model by using their a posteriori probabilities as weight information. When the controller is in a deadlock state, a deadlock avoidance strategy is given and the convergence of the corresponding a posteriori probability is proved. The simulation results show that, when the system is normal, the controller is the optimal LQG control, and, when faults occur in the system, the controller is able to track optimal control quickly, which enables the performance of the fault system to closely follow the performance of that under OC. In addition, the controller neither needs to detect the system fault model, nor needs to detect the fault time. Soft switching strategy is implemented among the multiple models, which avoids the jitter caused by frequent hard switching to the system. If the system noise is non-Gaussian, extending the above algorithm to a non-Gaussian stochastic system will be future work.

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